Формулы Фейнмана и интегралы по пространствам функций

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Representations of solutions of evolutionary differential equations.

$$\frac{\partial f}{\partial t} = \frac{1}{2}\triangle f + V(\cdot)f; \quad f : [0,\infty) \times E \to \mathbb{C}; \quad f(0,\cdot) = g$$

$$f(t,q) = \int_{C_0([0,t],E)} e^{\int_0^t V(\xi(\tau)+q)d\tau} g(\xi(\tau)+q)\nu_W(d\xi)$$

$$\nu_W\{\xi \in C_0([0,t],E); \xi(t_j) \in (a_j,b_j), j=1,2,...,n\} =$$
$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2}...\int_{a_n}^{b_n} \frac{p(t_1,0,q_1)}{\int_E p(t_1,0,q_1)dq_1}\frac{p(t_2-t_1,q_1,q_2)}{\int_E p(t_2-t_1,q_1,q_2)dq_2}\frac{p(t_3-t_2,q_2,q_3)}{\int_E p(t_3-t_2,q_2,q_3)dq_3}, ...,$$
$$\frac{p(t_4-t_3,q_3,q_4)}{\int_E p(t_4-t_3,q_3,q_4)dq_4} dq_1...dq_n =$$

$$= (\int_{E\times E\times...\times E} p(t_1,0,q_1)p(t_2-t_1,q_1,q_2)p(t_3-t_2,q_2,q_3), ...,$$
$$p(t_4 - t_3,q_3,q_4)dq_1...dq_n)^{-1}\int_{a_1}^{b_1}\int_{a_2}^{b_2}...\int_{a_n}^{b_n} p(t_1,0,q_1)p(t_2 -$$
$$t_1,q_1,q_2)p(t_3-t_2,q_2,q_3), ..., p(t_4-t_3,q_3,q_4)dq_1...dq_n;$$
$$p(t,q_1,q_2) = e^{-\frac{\|q_2-q_1\|^2}{2t}}$$

$$i\hbar\frac{\partial f}{\partial t} = -\frac{\hbar^2}{2m}\frac{1}{2}\triangle f + V(\cdot)f;\ \ f:[0,\infty)\times E\to\mathbb{C};\ \ f(0,\cdot)=g$$

$$\hbar = 1,\ m = 1$$

$$f(t,q) = \int_{C_0([0,t],E)} e^{-i\int_0^t V(\xi(\tau)+q)d\tau} g(\xi(\tau)+q)\Phi_W(d\xi)$$

Here $p(t,q_1,q_2) = e^{i\frac{\|q_2-q_1\|^2}{2t}}$

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Some definitions of Feynman pseudomeasures.

1. As a limit of integrals over Cartesian products of a space(Feynman, 1948 (configuration space), 1951 (phase space); Nelson, 1964 (configuration space)).

2. White noise analysis (Hida-Streit, 1975-80).

3. Analytical continuation (Gelfand-Yaglom, 1956, configuration space); Smolyanov-Shavgulidze, 1990, phase space).

4. Fourier transform (Cecile deWitt-Morette , 1974).

5. Parceval identity (Maslov-Chebotarev, 1976, Albeverio-Hoeg-Krohn, 1976).

6. Central limit theorem (Smolyanov-Khrennikov, 1985).

A Feynman formula A Feynman formula is a representation of either the Schrödinger group $e^{it\hat{H}}$ or the Schrödinger semigroup $e^{t\hat{H}}$ by limits of integrals over Cartesian powers of some space E (here H is a classical Hamilton function and $\hat{H}$ is a corresponding quantum mechanical Hamiltonian); if E is the phase (resp. configuration) space of the corresponding classical Hamiltonian system then one can speak of the Feynman formula in the phase (resp. configuration) space. The multiple integrals in Feynman formulae approximate integrals with respect to some measure or pseudomeasure on a set of functions which take values in E and are defined on a real interval (such functions are also called paths in E)

A Feynman-Kac formula Hence Feynman formulae imply some representations of Schrödinger groups and semigroups by integrals over paths in E; such representations are called Feynman-Kac formulae. A pseudomeasure is, by definition, a linear functional Φ on a suitable space of functions defined on the space of paths in E; the integral of a function f with respect to this pseudomeasure is defined as $\Phi(f)$. So deriving the Feynman formulas is the key step in evaluating path integrals which represent solutions to the Cauchy problem for the heat and Schrödinger equations. An important role is played by the Chernoff theorem which is a generalization of the well-known Trotter theorem.

Theorem (Chernoff) Let X be a Banach space. Let $F : [0, \infty) \to \mathcal{L}(X)$ be a strongly continuous mapping such that $F(0) = I$, $\|F(t)\| \le \exp(at)$ for some $a \in \mathbb{R}$, D be a linear subspace in $D(F'(0))$ and the restriction of $F'(0)$ to D be a closable operator whose closure we denote by C. If C is the generator of a strongly continuous semigroup $\exp(tC)$, then $F(t/n)^n$ converges to $\exp(tC)$ as $n \to \infty$ in the strong operator topology uniformly with respect to $t \in [0, T]$ for each $T > 0$.

Example: Trotter formula. Let $F(t) = e^{tA}e^{tB}$; then $F'(0) = A + B$, $F(0) = I$, $e^{t(A+B)} = \lim(e^{\frac{t}{n}A}e^{\frac{t}{n}B})^n$.

Applications 1. Equations on Riemannian manifolds.

2. Random fields in Riemannian manifolds.

3. Feynman formulae in phase spaces.

4. Equations in bounded domains.

5. Quasiparticles with position-dependent mass.

6. Selfadjoint equations of operators and the Feynman formulae.

7. Diffusion and quantum dynamics on graphs.

8.Equations on infinite dimensional manifolds of paths in Riemannian manifolds.

Quantum dynamics and diffusion on $[0,\infty)$. Let $\hat{H}$ is a perturbation of a selfadjoint extension of Δ $(= \frac{d^2}{dx^2})$ in $L_2(0,\infty)$. The collection $\mathcal{H}$ of such extensions is labelled by elements of $(-\infty,\infty]$. If $a \in (-\infty,\infty]$, then the selfadjoint extension $\hat{H}_a \in \mathcal{H}$ is defined as follows. Let $W = W(0,\infty)$ be the set of those differentiable functions on $[0,\infty)$ whose derivatives are abs. continuous and in turn have derivatives belonging to $L_2(0,\infty)$. Then $dom\hat{H}_a = \{f \in W : f(0) = af'(0)\}$ if $a \neq \infty$ and $dom\hat{H}_\infty = \{f \in W : f'(0) = 0\}$ if $f \in dom\hat{H}_a$, то $\hat{H}_a f(x) = f''(x)$. If V is a bounded continuous function on$(0,\infty)$, then the operator $f \mapsto \hat{H}_a f + V(\cdot) f$ with the same domain is also selfadjoint; we denote this operator by $\hat{H}_a + V$.

Let $X = L_2(0,\infty)$, $a \in (-\infty,\infty]$ and, for $t > 0$, the mappings $F_1(t) : X \to L_2(-\infty,\infty)$, $F_2(t) : L_2(-\infty,\infty) \to X$, $F_3(t) : X \to X$, $F^a(t) : X \to X$ are defined by:

$$F_1(t)(f)(x) = \frac{\frac{1}{2t}\int_0^{2t} f(z)dz(1+ax)}{1+at}\psi_t(x) + f(x)\varphi_t(x),$$

where $\varphi_t = \eta_t - \psi_t$, $\eta_t : \mathbb{R}^1 \to [0,1]$ is a smooth function such that $\eta_t(x) = 0$ for $x < -3t$, $\eta_t(x) = 1$ for $x > -2t$ and $\psi_t(x) = \eta_t(x)\eta_t(-x)$ for x ; $F_2(t)(f)(x) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} \exp(-\frac{(x-z)^2}{2t})f(z)dz$; $F_3(t)(f)(x) = e^{V(x)}f(x)$, $F^a(t) = F_3(t)F_2(t)F_1(\sqrt{t})$.

Theorem 1. For any $\varphi \in X$, $t > 0$ the following Feynman formula holds: $e^{t(\hat{H}_a+V)}\varphi = \lim_{n\mapsto\infty}\left(F^a\left(\frac{t}{n}\right)\right)^n\varphi$ в X.

Let $F_2^1(t),\ t\in(0,\infty)$ be a continuous map of $L_2(-\infty,\infty)$ into X whose restriction to $\mathcal{D}(-\infty,\infty)$ is defined by

$$F_2^1(t)(f)(x) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi t}}\int_{-\infty}^{\infty}\exp(i\frac{(x-z)^2}{2t})f(z)dz,$$

$F_3^1(t)(f)(x) = e^{iV(x)}f(x)$, $F^{1a}(t) = F_3^1(t)F_2^1(t)F_1(\sqrt{t})$.

Theorem 2. For any $\varphi\in L^2(0,\infty)$, $t>0$, the following Feynman formula holds: $e^{it(\hat{H}_a+V)}\varphi = \lim_{n\mapsto\infty}\left(F^{1a}\left(\frac{t}{n}\right)\right)^n\varphi$ в $L^2(0,\infty)$.

Feynman formulae for quantum dynamics and diffusion of particle with position-dependent mass. We consider perturbations of selfadjoint extensions of $\Delta_{g,0}$ in $L_2(-\infty,\infty)$. Here g is defined by: $g(x) = c_1 > 0$ for $x < 0$ and $g(x) = c_2 > 0$ for $x > 0$.

We consider selfadjoint extensions described by Gadella and his collaborators. Any such extensions is defined by an invertible operator $T = (t_{ij})$ in $\mathbb{C}^2$ as follows: the domain is the collection of functions $h \in W(-\infty,\infty)$ such that if $h^+ = (h^+(0), h'^+(0))(\in \mathbb{C}^2)$, $h^- = (h^-(0), h'^-(0))$, then $h^+ = Th^-$. This extension is denoted by $\hat{H}_T$.

Let $X = L_2(-\infty, \infty)$, T be the operator defined a selfadjoint extension and let for each $t > 0$, the mapings $F_4(t) : X \to X \oplus X, F_5(t) : X \oplus X \to X$, $F_6(t) : X \to X$, $F^T(t) : X \to X$ are defined as follows. $F_4(t)(f) = (h, k) \in X \oplus X$, where $h(x) = (a_+x + z_+)\psi_t(x) + f(x)\varphi_t$, $k(x) = (a_-x + z_-)\psi_t(x) + \varphi_t(-x)f(x)$ and the functions φ_t и ψ_t are the same as above and a_+, a_-, z_+, z_- are derined by $a_+t + z_+ = \frac{1}{2t}\int_0^{2t} f(z)dz$, $-a_-t + z_- = \frac{1}{2t}\int_{-2t}^{0} f(z)dz$ $z_+ = t_{11}z_- + t_{12}a_-$, $a_+ = t_{21}z_- + t_{22}a_-$. Moreover, $F_5(t)((h, k))(x) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} \exp(-\frac{(x-z)^2}{2t})h(z)dz$ for $x > 0$, $F_5(t)((h, k))(x) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} \exp(-\frac{(x-z)^2}{2t})k(z)dz$ for $x < 0$; $F_6(t)(f)(x) = e^{V(x)}f(x)$; $F^T(t) = F_6(t)F_5(t)F_4(\sqrt{t})$.

Theorem 3. For any $\varphi \in X$ $t > 0$ the following Feynman formula holds: $e^{t(\hat{H}_T + V)}\varphi = \lim_{n\mapsto\infty}\left(F^T\left(\frac{t}{n}\right)\right)^n \varphi$ в X.

Let $F_5^1(t),\ t \in (0, \infty)$ be a continuous mapping of — X into X, whose restriction to $\mathcal{D}(-\infty, \infty)$ is defined by:

$F_5^1(t)((h,k))(x) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi t}}\int_{-\infty}^{\infty}\exp(i\frac{(x-z)^2}{2t})h(z)dz$ if $x > 0$,

$F_5^1(t)((h,k))(x) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi t}}\int_{-\infty}^{\infty}\exp(i\frac{(x-z)^2}{2t})k(z)dz$ if $x < 0$

and let

$F_6^1(t)(f)(x) = e^{iV(x)}f(x),\ F^{1T}(t) = F_6^1(t)F_5^1(t)F_4(t)$.

Theorem 4. For any $\varphi \in X$, $t > 0$ the following Feynman formula holds: $e^{it(\hat{H}_T + V)}\varphi = \lim_{n\mapsto\infty}\left(F^{1T}\left(\frac{t}{n}\right)\right)^n \varphi$ в X.

Cauchy-Dirichlet problem. Let $\mathcal{G}$ be a domain in a Riemannian manifold K, having the smooth boundary Γ, let ν be the Borel measure in $\mathcal{G}$, generated by the volume formand let D be the differential operator, in the space of functions on $\mathcal{G}$, defined by $Df = \frac{1}{2}\Delta f + Vf$. One consider Cauchy–Dirichlet problems for the Schrödinger equation $if'(t) = D(f(t))$ and for the heat equation $f'(t) = D(f(t))$; here $f : [0, a) \to E = \mathcal{L}_2(\mathcal{G}, \nu), a > 0$. Let $\mathcal{D}_D$ be the selfadjoint operator in $\mathcal{G}$ corresponding to. We want to find $e^{it\mathcal{D}_D f}$ and $e^{t\mathcal{D}_D f}$), $t \in [0, a)$ for any $f \in C_0$ where C_0 is the collection of continuous functions on $\mathcal{G}$ vanishing on Γ. If $\psi(= \psi(\cdot))$ is a function on $\mathcal{G}$ then the mapping $\varphi \mapsto [\mathcal{G} \ni x \mapsto \psi(\cdot)\varphi(\cdot)]$ is also denoted by ψ.

Feynman formulae for Cauchy-Dirichlet problem.

Let $g_R(t,q_1,q_2) = e^{-\frac{\rho^2(q_1,q_2)}{2t}}$, $g(t,q_1,q_2) = e^{-\frac{\|q_2-q_1\|^2}{2t}}$, $\Phi(t,q_1,q_2) = e^{-\frac{i\|q_2-q_1\|^2}{2t}}$, $\|\cdot\|$ be the norm in $\mathbb{R}^n$, $q_j \in K$.

For $F : (0,\infty) \times K \times K \to \mathbb{C}$ and $t \in (0,\infty)$ let $A_F(t)$ be the operator in the space of complex functions on K, defined by: $(A_F(t)f)(q) = \int_K F(t,q,q_1)f(q_1)\nu(dq)$. The similar notation is used for operators in the space of functions on $\mathcal{G}$

Theorem. If $K = \mathbb{R}^n$ and $f \in C_0$ then for any $t > 0$

$$(e^{t\mathcal{D}_D}f)(q) = \lim_{n\to\infty}(((\sqrt{2\pi t/n})A_{g_R}(t/n)e^{\frac{t}{n}V(\cdot)})^n f)(q);$$

$$(e^{it\mathcal{D}_D}f)(q) = \lim_{n\to\infty}(((c_1(t/n)(\cdot)A_{\Phi}(t/n)e^{\frac{t}{n}iV(\cdot)})^n f)(q),$$

$c_1^{-1}(t) = \int_{\mathbb{R}^n}\Phi(t,q_1,q_2)dq_2$ (r.h.s. does not depend on q_1).

Theorem. If $f \in C_0$, $t > 0$ then:

$$(e^{t\mathcal{D}_D} f)(q) = \lim_{n\to\infty} c_2(t,n,q)(A_{g_R}(t/n)e^{\frac{t}{n}V(\cdot)})e^{\frac{t}{6n}K(\cdot)})^n f)(q),$$

where $(c_1(t,n,q))^{-1} = (A_{g_R}(t/n))^n \mathbf{1})(q)$ and $\mathbf{1}$ is the function on K whose values at each point are equal to 1.

$$(e^{t\mathcal{D}_D} f)(q) = \lim_{n\to\infty} c_3(t,n,q)(A_g(t/n)e^{\frac{t}{n}V(\cdot)})e^{\frac{t}{4n}K(\cdot)}e^{-\frac{t}{8n}a_K(\cdot)})^n f)(q),$$

where $(c_3(t,n,q))^{-1} = (A_g(t/n))^n \mathbf{1})(q)$.

The similar statements are valid for $e^{it\mathcal{D}_D}$.

Diffusion on graphs. Let $A = \{(a,0,0) : a \in [0,\infty\}$, $B = \{(0,a,0) : a \in [0,\infty)\}$, $C = \{(0,0,a) : a \in [0,\infty)\}$, $G = A \cup B \cup C$, $\hat{H}$ is a perturbation of a symmetrical closed extension of Δ $(= \frac{d^2}{dx^2})$ in $L_2(G)$. If V is a bounded continuous function on$(0,\infty)$, then the operator $f \mapsto \hat{H}f + V(\cdot)f$ is also symmetrical and closed; we denote this operator by $\hat{H} + V$.

$F_2(t)(f)(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp(-\frac{dist(x,z)^2}{2t}) f(z)dz$ where $x = (x_1, x_2, x_3), z = (z_1, z_2, z_3), dist(x,z)$ is the distance between x and z ; $F_3(t)(f)(x) = e^{V(x)} f(x)$, $F(t) = F_3(t)F_2(t)F_1(\sqrt{t})$, F_1 is similar to the function above for which a similar notation is used.

Theorem 1a. For any $\varphi \in X$, $t > 0$ the following Feynman formula holds: $e^{t(\hat{H}+V)}\varphi = \lim_{n \mapsto \infty} \left(F\left(\frac{t}{n}\right)\right)^n \varphi$ в X.

$F_2^i(t)(f)(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp(i\frac{dist(x,z)^2}{2t}) f(z)dz$ where $x = (x_1, x_2, x_3), z = (z_1, z_2, z_3), dist(x,z)$ is the distance between x and z ; $F_3^i(t)(f)(x) = e^{-iV(x)}f(x)$, $F^i(t) = F_3^i(t)F_2^i(t)F_1(\sqrt{t})$.

Theorem 2a. For any $\varphi \in X$, $t > 0$ the following Feynman formula holds: $e^{it(\hat{H}+V)}\varphi = \lim_{n \mapsto \infty} \left(F^i\left(\frac{t}{n}\right)\right)^n \varphi$ в X.

Problem. To define the measure and pseudomeasure in the Feynman-Kac formulae corresponding to the Feynman formulae in Theorems 1a and 2a.

Some historical remarks.

Configuration space	Phase space
Feynman, 1942-48	Feynman, 1951
Trotter-Daleckii, 1960-61	Chernoff, 1968
E.Nelson, 1964	S.–Tokarev–Truman, 2002

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