Quantization of Markov process

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To produce Markov process using quantum system one can alternate unitary evolution during time intervals $\Big((N-1)\delta t, N\delta t\Big)$ (here $N \in \mathbb{Z}$) and measurement of observable $\hat{A}$ at time moments $N\delta t$. Values of observable $\hat{A}$ in this series are described by some Markov process. We study the inverse problem to reconstruct the unitary quantum evolution between measurements from the known Markov process.

In the simplest formulation of the problem the Markov process with discrete time on a discrete state space is considered.

$$p(n, N\delta t) = \sum_{n'} M(n,n')\, p(n',(N-1)\delta t), \quad \sum_{n} M(n,n') = 1, \quad M(n,n') \geq 0.$$

Here $p(n, N\delta t) \in [0,1]$ is probability of state n at time $N\delta t$. Let states n are nondegenerate eigenstates of observable $\hat{A}$, e.g. $\hat{A} = \sum_n |n\rangle n \langle n|$, $\langle n|n'\rangle = \delta_{nn'}$. Measurement is considered to be projective.

We have to reconstruct the evolution operator matrix $\langle n|\hat{U}_{\delta t}|n'\rangle = U(n,n')$ from a transition matrix $M(n,n')$. Obviously

$$U(n,n') = \sqrt{M(n,n')}\, \mathrm{e}^{\mathrm{i}\alpha_{n,n'}}, \qquad \alpha_{n,n'} \in \mathbb{R}. \tag{1}$$

The only problem is to find the phase parameters $\alpha_{n,n'}$.

The problem involves the following questions

1. What are the conditions the problem has a solution (quantizability conditions)?
2. How to build $U(n,n')$ from a transition matrix $M(n,n')$?
3. What are the arbitrary parameters the matrix $U(n,n')$ depends on?
4. Which arbitrary parameters are related with the ambiguity of quantum system description (gauge parameters) and which are physically significant?

The necessary quantizability conditions are

$$\sum_{n'} M(n,n') = 1, \quad \forall n: \quad \sqrt{M(n_1,n)\cdot M(n_2,n)} \leq \sum_{n'\neq n} \sqrt{M(n_1,n')\cdot M(n_2,n')},$$

$$\forall n: \quad \sqrt{M(n,n_1)\cdot M(n,n_2)} \leq \sum_{n'\neq n} \sqrt{M(n',n_1)\cdot M(n',n_2)}.$$

Arbitrary parameters are related with phase factors. Each raw of matrix $U(n,n')$ could be multiplied by arbitrary phase factor $\mathrm{e}^{\mathrm{i}(\alpha_n+\beta_n)}$, and each column by $\mathrm{e}^{\mathrm{i}(\alpha_{n'}-\beta_{n'})}$. For matrix $D\times D$ we have $2D-1$ arbitrary parameters (multiplication of all matrix elements by the same factor could be done by two ways). Let $\sum_n \alpha_n = \sum_n \beta_n = 0$, then one has $2D-2$ independent α-s and β-s and one extra factor $\mathrm{e}^{\mathrm{i}\mu}$ for all matrix elements.

Gauge transformations are related with μ (shift of zero level of energy, 1 parameters) and β_n (multiplication of state $|n\rangle$ by phase factor $\mathrm{e}^{\mathrm{i}\beta_n}$, $D-1$ parameters).

The quantization of Markov process with continuous time on a discrete state space

$$\frac{d}{dt}p(n,t) = \sum_{n'} m(n,n')\,p(n',t), \quad \sum_{n} m(n,n') = 0, \quad \text{if } n\neq n',\ m(n,n')\geq 0.$$

is related with time rescaling. If $\delta t\to 0$ for fixed Hamiltonian $\hat{H}$ we produce quantum Zeno effect and evolution is frozen by frequent measurement. The rescaling is $t_{Markov} = \frac{t_{quantum}}{\delta t}$.

The necessary and sufficient condition of quantization is symmetry of transition matrix $m(n,n') = m(n',n)$. For nondiagonal matrix elements of Hamiltonian one has

$$H_{n,n'} = \langle n|\hat{H}|n'\rangle = \hbar\sqrt{m(n,n')}\,\mathrm{e}^{\mathrm{i}\alpha_{n,n'}}, \quad \alpha_{n,n'} = -\alpha_{n',n}\in\mathbb{R}, \quad n\neq n'. \tag{2}$$

The complete set of arbitrary real parameters are antisymmetric matrix $\alpha_{n,n'}$ ($\frac{D(D-1)}{2}$ independent parameters) and diagonal matrix elements of Hamiltonian (D parameters). One has D gauge parameters ($D-1$ relative phase factor for states $|n\rangle$ and 1 shift of zero energy level). The number of parameters above assumes that all matrix elements are nonzero $m(n,n')\neq 0$.

In the case of Markov process with continuous time and continuous state space

$$\frac{d}{dt}p(x,t) = \mathcal{L}[p](x,t).$$

We expect the time rescaling depends on properties of linear operator $\mathcal{L}$ and the model of measurement process. For differential operator rescaling could depend on order of differentiation.